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On the Independence Numbers of a Matroid

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Given a finite subset E of a vector space of dimension 4, the number of k-independent subsets of E will be denoted by I_k. We prove that $kI_k^2 \geqslant (k+1)I_{k-1}I_{k+1} + I_{k-1}I_k$. The equality holds if and only if all 4-subsets of E are independent. We prove this relation for matroids of rank 4. In particular we prove Mason's conjecture on the independence numbers of a matroid for $k=3$.

1. INTRODUCTION

We use the notations and terminology of Welsh [7].

Let M be a matroid. The set of independent k-sets will be denoted by $\mathscr{I}_k$ and $|\mathscr{I}_k|$ by I_k.

The following conjectures are well known.

Conjecture 1.1 (Welsh [7]). The sequence $\{I_k\}$ is unimodal.

Conjecture 1.2 (Mason [3]). $I_k^2 \geqslant I_{k+1}I_{k-1}$.

Conjecture 1.3 (Mason [3]). $k(n-k)I_k^2 \geqslant (k+1)(n-k+1)I_{k+1}I_{k-1}$, where n is the number of elements of M.

Conjecture 1.2 is implied by Dowling's partition conjecture [1] and by a property suggested by Stanley [6]. It is proved for $k \leqslant 7$ by Dowling [1]. Seymour proved Conjecture 1.3 for matroids without a circuit C such that $3 \leqslant |C| \leqslant k-1$ [4].

We propose the following conjecture.

Conjecture 1.4.

$$kI_k^2 \geqslant (k+1)I_{k-1}I_{k+1} + I_{k-1}I_k.$$

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Conjecture 1.4 implies Conjecture 1.3 with a strict inequality unless every $(k+1)$-subset is independent. This follows immediately from the obvious inequality $(k+1)I_{k+1} \leqslant (n-k)I_k$ which is strict unless every $(k+1)$-subset is independent.

2. Notations and Preliminairies

Let G be a graph and let x be a vertex of G. The set of vertices adjacent to x will be denoted by $\Gamma(x)$. The degree of x will be denoted by $d(x)$. All graphs considered are assumed to be without loops or multiple edges.

Let $G=(V, W; E)$ be a bipartite graph without isolated vertices and let $x \in V \cup W$. We define

$$\theta_G(x) = \sum_{y \in \Gamma(x)} (d(y))^{-1}.$$

Lemma 2.1 (Seymour [5]).

$$\sum_{X \in V} \theta(x) = |W|.$$

Proof.

$$\sum_{X \in V} \theta(x) = \sum_{X \in V} \sum_{y \in \Gamma(x)} (d(y))^{-1} = \sum_{y \in W} \sum_{x \in \Gamma(y)} (d(y))^{-1} = |W|. \quad \blacksquare$$

This lemma is stated by Seymour [5] in a particular case.

Let M be a matroid and let

$$E = \{\{X, Y\} \mid X \in \mathscr{I}_{k-1},\ Y \in \mathscr{I}_k \text{ and } X \subset Y\}.$$

The bipartite graph $(\mathscr{I}_{k-1}, \mathscr{I}_k; E)$ will be denoted by $G_k(M)$. We will write Γ_k (resp. θ_k) for Γ_{G_k} (resp. θ_{G_k}). The reference to M will be implicit in general. For $A \subset E(M)$, the matroid obtained by restriction to A (resp. contracting A) will be denoted by $M[A]$ (resp. M/A).

We see easily that $\theta_k(X) = d(X)/k$, for every $X \in \mathscr{I}_{k-1}$.

Lemma 2.2 (Seymour [5]). *Let $G=(V, W; E)$ be a bipartite graph such that for every $Y \subset W$ $|\Gamma(Y)|\,|W| \geqslant |Y|\,|V|$. Then there is a mapping $f\colon V \times W \to N$ such that $f(x, y)=0$ for any two non-adjacent vertices x and y and for each $y \in W$, $\sum_{x \in V} f(x, y) = |V|$ and for each $x \in V$, $\sum_{y \in W} f(x, y) = |W|$.*

This lemma is an easy generalization of the exercise in Seymour [5, p. 20, (2.4)].

LEMMA 2.3. *Let M be a matroid and let k be an integer, $1 \leqslant k \leqslant r(M)-1$. Let $(X, Y) \in \mathscr{I}_{k-1} \times \mathscr{I}_k$ be an edge of G_k. Then $k\theta_k(X) \geqslant (k+1)\theta_{k+1}(Y)+1$. Moreover the inequality is strict for at least one pair unless the matroid contains no circuit of order less than $k+2$.*

Proof. Let $\{b\} = Y - X$. For every $Z \in \Gamma_{k+1}(Y)$, let $f(Z) = Z - \{b\}$. The mapping f is clearly an injection from $\Gamma_{k+1}(Y)$ to $\Gamma_k(X) - \{Y\}$. Therefore $d_{k+1}(Y) \leqslant d_k(X) - 1$.

Since $\theta_k(X) = d(X)/k$ and $\theta_{k+1}(Y) = d(Y)/(k+1)$ we have $k\theta_k(X) \geqslant (k+1)\theta_{k+1}(Y)+1$.

Suppose that there is a circuit C of M with cardinality less than $k+2$. Let B be a basis such that $|B \cap C| = |C| - 1$. Let Y be a k-subset of B such that $|Y \cap C| = |C| - 1$ and $X = Y - b$, where $b \in C \cap Y$. Let $c \in C - Y$. It is clear that $X \cup \{c\} \in \Gamma_k(X) - f(\Gamma_{k+1}(Y))$. ∎

3. A STRONG PROPERTY

Let M be a matroid such that $E(M) = A \cup B$, where $A \in \mathscr{I}_{k-1}$ and $B \in \mathscr{I}_k$ and $A \cap B = \varnothing$. The complement of X will be denoted by $\bar{X}$.

We take $\mathscr{J}_{k-1} = \{X \in \mathscr{I}_{k-1} \mid \bar{X} \in \mathscr{I}_k\}$ and $\mathscr{J}_k = \{X \in \mathscr{I}_k \mid \bar{X} \in \mathscr{I}_{k-1}\}$.

The subgraph of $G_k(M)$ induced on $\mathscr{J}_{k-1} \cup \mathscr{J}_k$ will be denoted by $H_k(M)$. It is clear that $|\mathscr{J}_{k-1}| = |\mathscr{J}_k|$. We define $\pi_{A,B} = \{(S, T) \in \mathscr{I}_{k-1} \times \mathscr{I}_k \mid S \cup T = A \cup B \text{ and } S \cap T = A \cap B\}$.

It is clear that $\{\pi_{X,Y} \mid (X, Y) \in \mathscr{I}_{k-1} \times \mathscr{I}_k\}$ is a partition of $\mathscr{I}_{k-1} \times \mathscr{I}_k$.

LEMMA 3.1. *Let $(A, B) \in \mathscr{I}_{k-1} \times \mathscr{I}_k$, and let $\mathscr{P}$ and $\mathscr{X}$ be two subsets of $\mathscr{I}_{k-1}$ and $\mathscr{I}_k$, respectively. Let $F = A \cup B$. Then $\pi_{A,B} \cap (\mathscr{P} \times \mathscr{X}) = \pi_{A,B} \cap (\mathscr{P}_F \times \mathscr{X}_F)$, where $\mathscr{P}_F = \mathscr{P} \cap \mathscr{I}_{k-1}(M[F])$ and $\mathscr{X}_F = \mathscr{X} \cap \mathscr{I}_k(M[F])$.*

This lemma is obvious.

We introduce some notations. Let $\mathscr{X}$ be a set of subsets of E and $C \subset E$,

$$\mathscr{X}^C = \{X \in \mathscr{X} \mid C \subset X\}$$

$$\mathscr{X} - C = \{X - C \mid X \in \mathscr{X}\}$$

$$\mathscr{X} + C = \{X \cup C \mid X \in \mathscr{X}\}.$$

LEMMA 3.2. *Let $(A, B) \in \mathscr{J}_{k-1} \times \mathscr{J}_k$ and let $\mathscr{X}$ be a subset of $\mathscr{J}_k$ and $C = A \cap B$. Let $M' = M/C$. Then $\Gamma'_{k'}(\mathscr{X}^C - C) + C \subset \Gamma_k(\mathscr{X})$, where $k' = k - |C|$.*

This lemma is obvious.

PROPOSITION 3.3. *Let M be a matroid such that $H_j(N)$ has a perfect matching for every $j \leqslant k$ and every minor N of M with order $2j-1$. Then for any $\mathscr{X} \subset \mathscr{I}_k$, $I_k\,|\Gamma_k(\mathscr{X})| \geqslant I_{k-1}\,|\mathscr{X}|$.*

Proof. We shall prove the stronger property

$$|\pi_{A,B} \cap (\mathscr{I}_{k-1} \times \mathscr{X})| \leqslant |\pi_{A,B} \cap (\Gamma_k(\mathscr{X}) \times \mathscr{I}_k)|, \qquad \text{for all } (A, B) \in \mathscr{I}_{k-1} \times \mathscr{I}_k.$$

Suppose the contrary. We may assume that k is minimal such that there is a matroid M verifying the conditions of the proposition and $(A, B) \in \mathscr{I}_{k-1} \times \mathscr{I}_k$ such that $|\pi_{A,B} \cap (\mathscr{I}_{k-1} \times \mathscr{X})| > |\pi_{A,B} \cap (\Gamma_k(\mathscr{X}) \times \mathscr{I}_k)|$. We shall also choose $|E(M)|$ minimal. Take $F = A \cup B$, $C = A \cap B$, and $M' = M/C$. We prove the following.

(1) $E(M) = F$. Lemma 3.1 shows that $M[F]$ is also a counterexample. Therefore $E(M) = A \cup B$, by the minimality of the order of M.

(2) $C = \varnothing$. Suppose the contrary. Let $A' = A - C$, $B' = B - C$, $k' = k - |C|$, and $\mathscr{X}' = \mathscr{X}^C - C$. Since $H_{k'}(M')$ has a matching and $k' < k$

$$|\pi_{A',B'} \cap (\mathscr{I}'_{k'-1} \times \mathscr{X}')| \leqslant |\pi_{A',B'} \cap (\Gamma'_{k'}(\mathscr{X}') \times \mathscr{I}'_{k'})|.$$

It follows that

$$|\pi_{A',B'} \cap (\mathscr{I}'_{k'-1} \times \mathscr{X}') + (C, C)| \leqslant |\pi_{A',B'} \cap (\Gamma'_{k'}(\mathscr{X}') \times \mathscr{I}'_{k'}) + (C, C)|.$$

Therefore using Lemma 3.2,

$$|\pi_{A,B} \cap ((\mathscr{I}'_{k'-1} + C) \times \mathscr{X}^C)| \leqslant |\pi_{A,B} \cap (\Gamma_k(\mathscr{X}) \times \mathscr{I}_k)|.$$

But $\pi_{A,B} \cap ((\mathscr{I}'_{k'-1} + C) \times \mathscr{X}^C) = \pi_{A,B} \cap (\mathscr{I}_{k-1}(M) \times \mathscr{X})$. Let $(X, Y) \in \pi_{A,B} \cap (\mathscr{I}_{k-1}(M) \times \mathscr{X})$. Then $X \cap Y = C$. Hence $Y \in \mathscr{X}^C$. It follows that $(X - C, Y - C) \in \mathscr{I}'_{k'-1}(M') \times \mathscr{X}'$

$$|\pi_{A,B} \cap (\mathscr{I}_{k-1}(M) \times \mathscr{X})| \leqslant |\pi_{A,B} \cap (\Gamma_k(\mathscr{X}) \times \mathscr{I}_k)|, \qquad \text{a contradiction.}$$

(3) Now $|\pi_{A,B} \cap (\mathscr{I}_{k-1} \times \mathscr{X})| = |\mathscr{X} \cap \mathscr{J}_k|$ and

$$|\pi_{A,B} \cap (\Gamma_k(\mathscr{X}) \times \mathscr{I}_k)| = |\Gamma_k(\mathscr{X}) \cap \mathscr{J}_{k-1}|.$$

But it is easy to check that $\Gamma_k(\mathscr{X}) \cap \mathscr{J}_{k-1} = \Gamma_k(\mathscr{X} \cap \mathscr{J}_k) \cap \mathscr{J}_{k-1}$. The existence of a matching in $H_k(M)$ shows that $|\Gamma_k(\mathscr{X} \cap \mathscr{J}_k) \cap \mathscr{J}_{k-1}| \geqslant |\mathscr{X} \cap \mathscr{J}_k|$. It follows that

$$|\pi_{A,B} \cap (\mathscr{I}_{k-1} \times \mathscr{X})| \leqslant |\pi_{A,B} \cap (\Gamma_k(\mathscr{X}) \times \mathscr{I}_k)|, \qquad \text{a contradiction.} \quad \blacksquare$$

There are graphic matroids M such that $H_k(M)$ has no matching. The following example due to M. Las Vergnas improves our previous one.

EXAMPLE (Las Vergnas). Let G be a graph with edge set $\{12, 13, 14, 15, 23, 24, 25\}$. $H_4(\mathbb{C}(G))$ has no perfect matching.

PROPOSITION 3.4. *Let M be a matroid such that $H_j(N)$ has a perfect matching for every $j \leqslant k$ and every minor N of M with order $2j-1$. Then $kI_k^2 \geqslant I_{k-1}I_k + (k+1)I_{k-1}I_{k+1}$. Moreover the inequality is strict unless every $(k+1)$-set is independent.*

Proof. By Proposition 3.3, for any $\mathscr{X} \subset \mathscr{I}_k$, $I_k\,|\Gamma_k(\mathscr{X})| \geqslant I_{k-1}\,|\mathscr{X}|$. By Lemma 2.2, there is a mapping $f\colon \mathscr{I}_{k-1} \times \mathscr{I}_k \to N$ such that $f(X, Y) = 0$ if X is not contained in Y, $\sum_X f(X, Y) = I_{k-1}$, and $\sum_Y f(X, Y) = I_k$. Using Lemma 2.3, we obtain

$$kf(X, Y)\theta_k(X) \geqslant (k+1)f(X, Y)\theta_{k+1}(Y) + f(X, Y) \tag{1}$$

Therefore

$$k\sum_X\Big(\sum_Y f(X, Y)\Big)\theta_k(X) \geqslant (k+1)\sum_Y\Big(\sum_X f(X, Y)\Big)\theta_{k+1}(Y) + \sum_X\sum_Y f(X, Y).$$

By Lemma 2.1 and Lemma 2.2, we obtain

$$kI_k^2 \geqslant (k+1)I_{k-1}I_{k+1} + I_{k-1}I_k.$$

If M contains a circuit of order $\leqslant k+1$, Lemma 2.3 shows that (1) is strict for a least one pair. ∎

4. A SPECIAL CASE

LEMMA 4.1. *Let M be a matroid such that $E(M)$ is the disjoint union of a k-independent set and a $(k-1)$-independent set. Then $H_k(M)$ has a perfect matching if and only if for every $\mathscr{X} \subset \mathscr{I}_{k-1}$*

$$|\{Y \in \Gamma_k(\mathscr{X}) \mid \bar{Y} \in \mathscr{I}_{k-1} - \mathscr{X}\}| \geqslant |\{X \in \mathscr{X} \mid \bar{X} \in \mathscr{I}_k - \Gamma_k(\mathscr{X})\}|.$$

Proof. By Hall's condition, there is a perfect matching if and only if for every $\mathscr{X} \subset \mathscr{I}_{k-1}$,

$$\begin{aligned}|\{Y \in \Gamma_k(\mathscr{X}) \mid \bar{Y} \in \mathscr{I}_{k-1} - \mathscr{X}\}| + |\{Y \in \Gamma_k(\mathscr{X}) \mid \bar{Y} \in \mathscr{X}\}| \\ \geqslant |\{X \in \mathscr{X} \mid \bar{X} \in \mathscr{I}_k - \Gamma_k(\mathscr{X})\}| + |\{X \in \mathscr{X} \mid \bar{X} \in \Gamma_k(\mathscr{X})\}|.\end{aligned}$$

But $|\{Y \in \Gamma_k(\mathscr{X}) \mid \bar{Y} \in \mathscr{X}\}| = |\{X \in \mathscr{X} \mid \bar{X} \in \Gamma_k(\mathscr{X})\}|$. ∎

LEMMA 4.2. *Let M be a matroid such that $E(M)$ is the disjoint union of a k-independent set and a $(k-1)$-independent set. Suppose that there is an element a of M contained in all k-independent sets. Then $H_k(M)$ has a perfect matching.*

Proof. Let X be an element of $\mathscr{I}_{k-1}(M)$. Then $a \in \bar{X}$. It is easy to see that $X \cup \{a\}$ is a k-independent set containing X. Moreover $\overline{X \cup \{a\}} = \bar{X} - a$. Hence $X \cup \{a\} \in \mathscr{I}_k(M)$. It follows that $\{(X, X \cup \{a\}); X \in \mathscr{I}_{k-1}(M)\}$ is a perfect matching. ▮

LEMMA 4.3. *Let M be a matroid such that $E(M)$ is the disjoint union of a k-independent set and a $(k-1)$-independent set. Suppose that $2 \leqslant k \leqslant 3$. Then $H_k(M)$ has a perfect matching.*

Proof. If $k = 2$ then the matroid is of order 3. If every 2-set is independent, the perfect matching is obvious. Otherwise there is an element contained in all 2-independent sets. The matching exists by Lemma 4.2.

Suppose the result false for $k = 3$. The matroid is of order 5. There is no element contained in all 3-independent sets, by Lemma 4.2. Therefore every 2-independent set is contained in at least two distinct 3-independent sets. Take $\mathscr{X}_1 = \{Y \in \Gamma_k(\mathscr{X}) \mid \bar{Y} \in \mathscr{I}_{k-1} - \mathscr{X}\}$ and $\mathscr{X}_2 = \{X \in \mathscr{X} \mid \bar{X} \in \mathscr{I}_k - \Gamma_k(\mathscr{X})\}$. The non-existence of a perfect matching and Lemma 4.1 show that $|\mathscr{X}_1| < |\mathscr{X}_2|$. Consider the bipartite graph R defined on $\mathscr{X}_1 \cup \mathscr{X}_2$ by inclusion. We have seen that every $X \in \mathscr{X}_2$ is contained in two 3-independent sets $X \cup \{a\}$ and $X \cup \{b\}$. Suppose that $X \cup \{a\} \notin \mathscr{X}_1$. Then $\bar{X} - a \in \mathscr{X}$. It follows that $\bar{X} \in \Gamma_k(\mathscr{X})$, contradicting the definition of $\mathscr{X}_2$. Hence $d(X) \geqslant 2$ in R. An easy argument shows that $\mathscr{X}_1$ contains at least two vertices of degree 3. The neighbourhood of these sets contains two disjoint sets $U, V \in \mathscr{X}_2$. Now $U \subset \bar{V}$. Hence $\bar{V} \in \Gamma_k(\mathscr{X})$, contradicting the definition of $\mathscr{X}_2$. ▮

PROPOSITION 4.4. *Let M be a matroid and k be an integer $\leqslant 3$. Then $kI_k^2 \geqslant I_{k-1}I_k + (k+1)I_{k-1}I_{k+1}$. Moreover the inequality is strict unless every $(k+1)$-set is independent.*

Proof. This is an easy application of Proposition 3.4 and Lemma 4.3. ▮

COROLLARY 4.5. *Let M be a matroid of order n. Then $I_3^2 \geqslant (4(n-2)/3(n-3))I_2I_4$. Moreover the inequality is strict unless the matroid contains no circuit of order $\leqslant 4$.*

The proof results from the remark given after Conjecture 1.4.

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